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PICARD GROUPS IN ABELIAN GROUP RINGS

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Let A be a commutative ring with identity. In the foundational paper of Bass and Murthy [4] techniques were developed for computing $\mathrm{Pic}(A\pi)$ (and related groups) for π a finitely generated abelian group. The procedure was to decompose π as $\pi_0 \times T$, where π_0 is finite and T is free abelian, so that $A\pi = R[T]$, $R = A\pi_0$. Then in the case that R is one-dimensional, Noetherian, and has finite integral closure, $\mathrm{Pic}(R[T])$ can be determined in terms of $\mathrm{Pic}(R)$ and certain invariants of R [4, Thm. 8.1]. For more general rings R it is still of interest to know when the canonical map $\mathrm{Pic}(R) \to \mathrm{Pic}(R[T])$ is an isomorphism. It turns out that the condition that this map be an isomorphism is a stronger condition on a ring R for T a free abelian group, than for T a free abelian monoid. In the latter case, R is said to be seminormal. The purpose of this note is to give a result on when a group ring $A\pi$ is seminormal, where π is a finite abelian group, and some results on when the map $\mathrm{Pic}(A) \to \mathrm{Pic}(A[X, X^{-1}])$ is an isomorphism for A an integral domain.

In Section one of this paper, we give several preliminary results on seminormality. Theorem 1.3 is of particular interest here for it allows us to show that a projective ideal I of $R[T]$, T a free abelian group or monoid, is extended from R by showing that $IR_\lambda[T]$ is extended from R_λ for certain families $\{R_\lambda\}_{\lambda \in \Lambda}$ of R-algebras.

In Section two we give our results on seminormality of group rings $A\pi$, π a finite abelian group. Let n be the order of π. Under the conditions that the total quotient ring of A is absolutely flat and nA has no embedded prime divisors, it is shown that $A\pi$ is seminormal if and only if A is seminormal, n is regular on A, and $nA, n\bar{A}$ are radical ideals, where $\bar{A}$ = integral closure of A. This result was first obtained by Bass and Murthy [4] for $A = Z$. It was extended to one dimensional Mori rings by Pedrini [20, Thm. 2] and to all Mori rings by Greco [9, Thm. 3.1], where by a Mori ring it is meant a reduced Noetherian ring R whose integral closure $\bar{R}$ is a finite R-module [9].

In Section three we consider the question of when the canonical map $\psi : \mathrm{Pic}(R) \to \mathrm{Pic}(R[T])$ is an isomorphism for T a free abelian group. Very few results on when this map is an isomorphism are known if R is not either normal, or a one-dimensional Mori ring. In Theorem 3.2 it is shown that if T has rank one, then ψ is an isomorphism if R is a seminormal domain in which the weak Bourbaki prime

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ideals of principal ideals are unibranched. For one dimensional Mori domains, these conditions are known to also be necessary [9, Thm. 4.5].

All rings are assumed to be commutative with identity. If R is a ring we let $T(R)$ denote the total quotient ring of R and $\bar{R}$ the integral closure of R in $T(R)$. If G is a free abelian group or monoid and $f \in R[G]$, we let $c_R(f)$ denote the ideal of R generated by the coefficients of f, and if $I \subseteq R[G]$, we let $c_R(I)$ = the ideal of R generated by $\{c_R(f) \mid f \in I\}$. We sometimes write $c(f)$ or $c(I)$ if the reference to R is clear. A prime ideal P of R is called a B_w-prime (weak-Bourbaki associated prime [5, p. 289, Exercise 17]) of an ideal I if for some $x \in R$, P is minimal among prime ideals containing $(I :_R x)$, and an N-prime (prime divisor in the sense of Nagata [17, p. 19]) of I if there is a multiplicatively closed subset S of R which does not meet I such that PR_S is maximal among the ideals of R_S which consist of zero divisors on R_S/IR_S. See [12], [32] for some relationships among B_w-primes, N-primes, and some other types of associated primes. Our main references for semi-normality are [30], [31].

Section 1

Our first lemma in this section is an immediate consequence of the special case in [9, Lemma 2.7]. (See [19] for some related results.) We will say that an integer m is regular on a ring R if multiplication by m is an injective map on R.

Lemma 1.1. *Let R be a ring and π an abelian group of order m. Then $R\pi$ is reduced if and only if R is reduced and m is R-regular.*

Proof. Write $R = \varinjlim R_\alpha$ where each R_α is a subring of R which is finitely generated as a Z-algebra. Then each R_α is reduced, Noetherian, and pseudo-geometric (=Nagata ring) [15, 31H, Thm. 72]. It follows that each R_α has finite integral closure and so we may apply [9, Lemma 2.7] to the R_α.

If $R \to B$ is a ring homomorphism, we will say that a B-module M is *extended from* R if $M \cong N \otimes_R B$ for some R-module N. Throughout this section $R[X]$ denotes the polynomial ring over R in an arbitrary set X of variables, unless otherwise noted.

Lemma 1.2. *Let R be a reduced ring and I an invertible ideal of $R[X]$ such that $I \cap R = I_0$ contains a regular element. If I is extended from R as an $R[X]$*-module, then $I = I_0R[X]$.

Proof. It suffices to show that $IR_p[X] = I_0R_p[X]$ for each prime ideal p of R, and since $IR_p[X] \cap R_p = I_0R_p[X]$ and I_0R_p contains a regular element, we may assume R is quasi-local. But then I is principal since I is extended from R. Let g be a generator of I and let $K=T(R)$. Since $IK[X]=K[X]$, g is a unit of $K[X]$ and $g\in K$ since K is reduced. Thus $g\in K\cap I=I_0$.

The following result and its corollaries are generalizations of [30, Lemma 6.4] on patching of rank one projective modules. It will be used in each of the following two sections. In the applications, we will have $K=T(R)$ absolutely flat, and such rings are easily seen to be seminormal and thus satisfy $\mathrm{Pic}(K[X])=0$.

Theorem 1.3. *Let R be a ring and let $\{\varphi_\lambda : R\to R_\lambda\}_{\lambda\in\Lambda}$ be a family of commutative reduced torsion-free R-algebras such that the induced map $\varphi : R\to B=\prod_{\lambda\in\Lambda} R_\lambda$ is injective and $B/\varphi(R)$ is R-torsion-free. If I is an invertible ideal of $R[X]$, which contains a regular element of R, and $IR_\lambda[X]$ is extended from R_λ for each $\lambda\in\Lambda$, then I is extended from R.*

Proof. *Let* $I_0=I\cap R$. Since regular ideals of $R[X]$ are generated by regular elements [16], then I is an intersection of principal fractional ideals [26, Prop. 3.1]. Thus as in [22, Lemma 1] we have that $I_0=\bigcap\{rc_R(g)^{-1}\mid I\subseteq (r/g)R[X], r\in R, g\in R[X]$ regular elements of $R[X]\}$. Since R_λ is R-torsion-free then $IR_\lambda[X]$ is an invertible ideal of $R_\lambda[X]$ which contains a regular element of R_λ. Therefore $IR_\lambda[X]=J_\lambda R_\lambda[X]$ where $J_\lambda=IR_\lambda[X]\cap R_\lambda$ by assumption and Lemma 2. Thus if $r\in R, g\in R[X]$ are regular elements with $I\subseteq (r/g)R[X]$ then

$$\begin{aligned} c_{R_\lambda}(g)c_{R_\lambda}(IR_\lambda[X]) &= c_{R_\lambda}(g)c_{R_\lambda}(J_\lambda R_\lambda[X]) = c_{R_\lambda}(g)J_\lambda \\ &= c_{R_\lambda}(gJ_\lambda R_\lambda[X]) = c_{R_\lambda}(gIR_\lambda[X]) \\ &\subseteq c_{R_\lambda}(rR_\lambda[X]) = rR_\lambda. \end{aligned}$$

But this gives $\varphi_\lambda(c_R(g)c_R(I))\subseteq \varphi_\lambda(r)R_\lambda$ for every λ, and thus since φ is injective and $B/\varphi(R)$ is R-torsion-free, it follows that $c_R(g)c_R(I)\subseteq rR$, and thus $c(I)\subseteq I_0$.

Remark 1.4. Let $\mathrm{NPic}(R)$ be the co-kernel of the map $\mathrm{Pic}(R)\to \mathrm{Pic}(R[X])$. If the R_λ in Theorem 1 are R-flat and $\mathrm{Pic}(K[X])=0$ where $K=T(R)$, then the conclusion of the above theorem is that the canonical map $\psi : \mathrm{NPic}(R)\to \prod_{\lambda\in\Lambda}\mathrm{NPic}(R_\lambda)$ is injective.

Proof. Let $M\in \mathrm{Pic}(R[X])$ represent an element of $\ker\psi$ and let S be the set of regular elements of R. From the exact sequence

$$U(K[X])\to \mathrm{Pic}(R[X],S)\to \mathrm{Pic}(R[X])\to \mathrm{Pic}(K[X])=0$$

[3, p. 136], we see that M is isomorphic to an ideal I of $R[X]$ with $I\cap S\neq\emptyset$. Further,

$$IR_\lambda[X]\cong I\otimes_{R[X]} R_\lambda[X]\cong M\otimes_{R[X]} R_\lambda[X],$$

the first isomorphism holding since R_λ is R-flat. Thus by the above theorem, I is extended from R, and hence M is also.

Corollary 1.5. *Let R be a reduced ring whose total quotient ring K satisfies $\mathrm{Pic}(K[X])=0$, and let $M\in\mathrm{Pic}(R[X])$. If M_p is extended from R_p for each B_w-prime of a principal ideal, then M is extended from R.*

Proof. Let $\mathscr{B}$ be the set of B_w-prime ideals of principal ideals. By Theorem 1.3 it suffices to show that the canonical map $\varphi: R\to B=\prod_{p\in\mathscr{B}} R_p$ is injective and that $B/\varphi(R)$ is R-torsion-free. If $a\in\ker\varphi$ and $a\neq 0$, let $p\in\mathrm{Spec}(R)$ be minimal over $(0:_R a)$. Then $p\in\mathscr{B}$ and $(0:_R a)R_p=(0:_{R_p} a/1)\subsetneq pR_p$, a contradiction. Thus φ is injective. To show that $B/\varphi(R)$ is R-torsion-free let $\mathscr{B}=\{p_\lambda \mid \lambda\in\Lambda\}$, let $r\in R$ be a regular element and let $(b_\lambda)_{\lambda\in\Lambda}\in B$ be such that $r(b_\lambda)_{\lambda\in\Lambda}=\varphi(a)$, $a\in R$. Then $a\in rR_p$ for every $p\in\mathscr{B}$ and hence $a\in rR$, say $a=ra'$, $a'\in R$. But since r is regular, $\varphi_\lambda(a')=b_\lambda$ for all $\lambda\in\Lambda$. Hence $B/\varphi(R)$ is R-torsion-free.

In [30, p. 217] a family $\{R_\lambda\}_{\lambda\in\Lambda}$ of R-algebras was said to be faithfully flat if each R_λ is R-flat, and for each R-module M, $R_\lambda\otimes_R M=0$ for all $\lambda\Rightarrow M=0$. The following result generalizes [30, Prop. 3.9]. An important special case, given in [10, Corol. 1.7] for Mori rings, says that if $T(R)$ is absolutely flat, and a faithfully flat R-algebra B is seminormal, then R is seminormal. A more direct argument for this case however is to observe that $R=B\cap T(R)$ by faithful flatness, and an intersection of seminormal rings is seminormal [30, Corol. 3.3].

Corollary 1.6. *Let R be a ring with $\mathrm{Pic}(K[X])=0$ where $K=T(R)$ and let $\{\varphi_\lambda: R\to R_\lambda\}_{\lambda\in\Lambda}$ be a faithfully flat family of commutative reduced R-algebras. Then the canonical map $\psi:\mathrm{NPic}(R)\to\prod_{\lambda\in\Lambda}\mathrm{NPic}(R_\lambda)$ is injective.*

Proof. This is immediate from Theorem 1.3 and Remark 1.4.

Corollary 1.7. *Let R be a reduced ring with $T(R)=K$ and let I be an invertible ideal of $R[X]$ such that $IK[X]=K[X]$. If R is an intersection of a family $\{R_\lambda\}_{\lambda\in\Lambda}$ of overrings of R such that $IR_\lambda[X]$ is extended from R_λ for each $\lambda\in\Lambda$, then I is extended from R.*

Proof. Let $\varphi: R\to B=\prod_{\lambda\in\Lambda} R_\lambda$ be the map induced by the inclusions. Then φ is easily seen to satisfy the hypothesis of Theorem 1.3.

In Section 3 we will need the following analogue of Corollary 1.7 where the free monoid algebra $R[X]$ is replaced by $R[T]$, T a free abelian group.

Theorem 1.8. *Let R be an integral domain with quotient field K, let T be a free abelian group and let I be an invertible ideal of $R[T]$ such that $I\cap R\neq\{0\}$. If R is an intersection of a family $\{R_\lambda\}_{\lambda\in\Lambda}$ of overrings such that $IR_\lambda[T]$ is extended from R_λ for each $\lambda\in\Lambda$, then I is extended from R.*

For the proof of this result we will just indicate the modifications needed in order to adapt the proof of Theorem 1.3 to this case. First we need the following analogue of Lemma 1.2. The proof is the same except that one uses the fact that the units of $K[T]$, where K is a field and T a free abelian group, are of the form ut with $u \in K - \{0\}$ and $t \in T$ [4, Prop. 5.12]. (See [8] for a further study of units in semigroup rings.)

Lemma 1.9. *Let R be an integral domain with quotient field K and let T be a free abelian group. If I is an invertible ideal of $R[T]$ with $I \cap R = I_0 \neq \{0\}$, and I is extended from R, then $I = I_0 R[T]$.*

To complete the proof of Theorem 1.8, we just observe that the proof given for Theorem 1.3 works for any content R-algebra and that $R[T]$ is a content R-algebra [18], [27]. Recall that an R-module M is called a *content R-module* if $x \in c(x)M$ for every $x \in M$ where $c(x)$ is defined as $\bigcap \{I \mid I$ is an ideal of R with $x \in IM\}$. An R-algebra R' is a *content R-algebra* if R' is a faithfully flat content R-module and for each $f, g \in R'$ there exists an integer $n \geq 0$ such that $c(fg)c(g)^n = c(f)c(g)^{n+1}$.

In [29] Swan has given an example of a rank 2 projective module P over $A[X, X^{-1}]$, where X is an indeterminate and $A = \mathbb{C}[X_0, \ldots, X_4]/(1 - \sum_{i=0}^{4} X_i^2)$, which is not extended from A but P_M is extended from $A_M[X, X^{-1}]$ for each maximal ideal M of A. Theorem 1.8 shows that one cannot get a similar example with P of rank one.

Lemma 1.10. *Let $\varphi: R \to B$ be étale. If R is seminormal, then B is seminormal.*

Proof. By [24, p. 51], for each prime ideal P of B there exists $b \in B - P$ and $r \in R - p$, where $p = P \cap R$, such that B_b is R isomorphic to $(R_r[X]/(f))_g$ where X is an indeterminate, $g, f \in R_r[X]$ and f' (=the derivative of f) is a unit of $(R_r[X]/(f))_g$. To show B is seminormal it suffices to show that each B_b is seminormal, and thus we are reduced to considering the standard étale algebras $R \to (R[X]/(f))_g$. By [30, Lemma 6.3] we may write R as a direct limit of seminormal subrings R_α which are finitely generated as Z-algebras and thus Mori. But then $(R[X]/(f))_g$ is a direct limit of algebras which are of the form $(R_\alpha[X]/(f))_g$ and thus are seminormal by [9, 1.6(ii)]. Therefore $(R[X]/(f))_g$ is seminormal [30, Lemma 6.2];

Corollary 1.11. *If a quasi-local ring R is seminormal, then its henselization R^{h} and strict henselization R^{sh} are seminormal.*

Proof. R^{h} and R^{sh} are direct limits of étale R-algebras [24, Chap VIII] and Pic(R) commutes with direct limits [30, Lemma 6.2].

The proof of the following result follows the outline of the proof given in [25, p. 82] of a classical result in number theory.

Lemma 1.12. *Let R be an integrally closed integral domain of characteristic zero, and let p be a prime integer such that $pR \neq R$ is a radical ideal of R. Then $R[X]/(X^{p-1}+X^{p-2}+\cdots+X+1)$ is an integrally closed domain, where X is an indeterminate.*

Proof. Let $f=X^{p-1}+X^{p-2}+\cdots+X+1$ and let K be the quotient field of R. If $P\in \operatorname{Spec}(R)$ is minimal over pR, then $f(X+1)$ is irreducible in $R_P[X]$ by [17, 31.11] and so is irreducible in $K[X]$ since R_P is integrally closed. Thus $R[X]/(f)$ is integral domain. Then $R[X]/(f)=R[x]$ where x is the image of X, $K[x]=S^{-1}R[x]$ is the quotient field of $R[x]$ where $S=R-\{0\}$, and $K[x]$ is a separable normal extension of K. Let A be the integral closure of R in $K[x]$. To show $A\subseteq R[x]$, let

$$y=a_0+a_1x+\cdots+a_{p-2}x^{p-2}\in A, \quad a_i\in K.$$

Then

$$xy=a_0x+a_1x^2+\cdots+a_{p-2}x^{p-1},$$

and subtracting we get

$$y(1-x)=a_0(1-x)+a_1(x-x^2)+\cdots+a_{p-2}(x^{p-2}-x^{p-1}).$$

Since $f(X)=\prod_{i=1}^{p-1}(X-x^i)$, $x,x^2,\ldots,x^{p-1}$ are conjugate and so have the same trace. Thus $\operatorname{Tr}(y(1-x))=\operatorname{Tr}(a_0(1-x))=a_0\operatorname{Tr}(1-x)=a_0p$. Let $y=y_1,y_2,\ldots,y_{p-1}$ be the conjugates of y. Then

$$\begin{aligned}\operatorname{Tr}(y(1-x))&=y_1(1-x)+y_2(1-x^2)+\cdots+y_{p-1}(1-x^{p-1})\\&=y_1(1-x)+y_2\left(\frac{1+x^2}{1-x}\right)(1-x)+y_3\left(\frac{1-x^3}{1-x}\right)(1-x)\\&\quad+\cdots+y_{p-1}\left(\frac{1-x^{p-1}}{1-x}\right)(1-x)\\&=(1-x)y'\in A(1-x).\end{aligned}$$

Thus $\operatorname{Tr}(y(1-x))\in A(1-x)\cap R$. But since $1-x^i$ and $1-x^j$ are associates for $i,j\in\{1,2,\ldots,p-1\}$,

$$[A(1-x)]^{p-1}=\prod_{i=1}^{p-1}A(1-x^i)=Af(1)=pA,$$

and since pR is a radical ideal, pR is contracted from A, so $pR=pA\cap R$. Thus $[A(1-x)]^{p-1}\cap R=pA\cap R=pR$, and since pR is a radical ideal, $A(1-x)\cap R=pR$. Therefore $a_0p=\operatorname{Tr}(y(1-x))\in pR\Rightarrow a_0\in R$. To show $a_j\in R$ multiply

$$y=a_0+a_1x+\cdots+a_{p-2}x^{p-2}$$

by x^{p-j} and get

$$\begin{aligned}yx^{p-j}&=a_0x^{p-j}+a_1x^{p-j+1}+\cdots+a_{j-1}x^{p-1}\\&\quad+a_j+a_{j+1}x+a_{j+2}x^2+\cdots+a_{p-2}x^{p-j-2}.\end{aligned}$$

Then by what we have shown, $a_j\in R$.

Lemma 1.13. *Let R, m be a seminormal quasi-local ring with $m = pR$, p a regular element of R. Let $R' \neq R$ be an overring of R which is a finite R-module, and let $b = (R :_R R')$. Then p is not a zero divisor on R'/b.*

Proof. Let $q = \bigcap_{i=1}^{\infty} p^i R$. Then q is a prime ideal of R and R/q is a principal valuation ring [2, Thm. 2.2]. Since b is a radical ideal of R [31, Lemma 1.3], we have either $b = m$ or $b \subsetneq q$ [2, Thm. 2.2]. Let $Q_1, \ldots, Q_s$ be the prime ideals of R' which lie over q, and consider the embedding $R/q \to R'/\bigcap Q_i$. Since $p \notin \bigcup_{i=1}^{s} Q_i$, p is not a zero divisor on $R'/\bigcap Q_i$. Thus the regular elements of R/q remain regular in $R'/\bigcap Q_i$ and therefore $T(R/q) \subseteq T(R'/\bigcap Q_i)$. If $b = m$, then $pR' \subsetneq R$; we get $T(R/q) = T(R'/\bigcap Q_i)$ and so $s = 1$. Since R/q is integrally closed and R'/Q_1 is integral over R/q, we get $R/q = R'/Q_1$ and therefore $R' = R + Q_1$. But since $Q_1 = \sqrt[R']{qR'} \subseteq \sqrt[R']{b} = b$, we get $Q_1 \subseteq R$ and $R' = R$, a contradiction. Thus $b \neq m$, and so $b \subsetneq q$.

Now to show that p is regular on R'/b it suffices to show that p is not contained in any B_w-prime of b [5, p. 289, Exer. 17(b)]. Since $pR = m$ the only primes of R' containing p are maximal ideals, so it suffices to show that no maximal ideal of R' is a B_w-prime of b. If M is a maximal ideal of R' which is minimal over $(b :_{R'} a)$ for $a \in R'$, then $MR'_M = \sqrt[R'_M]{(b :_{R'_M} a)} = b :_{R'_M} a$ by [32, Prop. 2.3]. Therefore M is a minimal prime divisor of b [32, Thm. 2.1]. But this contradicts the fact that $b \subsetneq q \subsetneq \bigcap_{i=1}^{s} Q_i$, and finishes the proof.

Section 2

We will now give our main result on seminormality of group rings. A partial converse follows from the result [9, Thm. 3.1] of Greco and will be given later in this section.

Theorem 2.1. *Let A be a seminormal ring with $T(A)$ absolutely flat and let π be a finite abelian group of order n. If n is regular on A, nA and $n\bar{A}$ are radical ideals, and nA has no embedded prime divisors, then $A\pi$ is seminormal.*

Proof. It suffices to show that $(A\pi)_Q$ is seminormal for every B_w-prime Q of a principal ideal by Corollary 1.5, and to show that $(A\pi)_Q$ is seminormal it suffices to show $A_P\pi$ is seminormal where $P = Q \cap A$. If $nA_P = A_P$, then $A_P\pi$ is étale over A_P [9, Corol. 2.5], and hence seminormal by Lemma 1.10. Otherwise we get $n = p^t m$ where $p \in P$ is prime and $m \notin P$. Then $nA_P = p^t A_P$, and $t = 1$ since nA is a radical ideal. Further, since Q is a B_w-prime of a principal ideal and $p \in Q$, we have by [6, Thm. 3] that Q is a B_w-prime of $pA\pi$, and thus by [12, Prop. 4.5] P is an N-prime of pA.

By hypothesis P is minimal over pA and so $pA_P = PA_P$ since pA_P is a radical ideal. Further $T(A_P) = T(A)_P$ is absolutely flat [7, Prop. 2]. Therefore we may assume that A, m is quasi-local with $m = pA$. Write $\pi = \pi' \times \pi''$ where $(\pi' : 1) = p$ and

$(\pi'':1)=m$. Then $A\pi=(A\pi')\pi''$ is étale over $A\pi'$ since $m(A\pi')=A\pi'$ [9, Corol. 2.5]. Thus by Lemma 1.10, $A\pi$ is seminormal if $A\pi'$ is. Therefore we are reduced to considering the case $(\pi:1)=p$. Now to show that $A\pi$ is seminormal we first show that $A\pi$ is seminormal in $\bar{A}\pi$. For this it suffices to show that $A\pi$ is seminormal in $A'\pi$ for each overring A' of A which is finite as a module over A. Let A' be such an overring and let $b=(A:_A A')$. Since A is seminormal in A', b is a radical ideal of A' [31, Lemma 1.3]. Further, $bA\pi$ is the conductor of $A\pi$ in $A'\pi$ and $A'\pi/bA'\pi\cong(A'/b)\pi$, and since p is a regular element on A'/b by Lemma 1.13, then $(A'/b)\pi$ is reduced by Lemma 1.1. Therefore $bA\pi$ is a radical ideal of $A'\pi$ and hence $A\pi$ is seminormal in $A'\pi$.

It remains to show that $\bar{A}\pi$ is seminormal by [30, Corol. 3.4]. For this it suffices to show that $(\bar{A})_P\pi$ is seminormal for every maximal ideal P of $\bar{A}$ [30, Prop. 3.7] and since $T(A)$ is absolutely flat, $(\bar{A})_P$ is an integrally closed domain [7, Props. 5 and 6]. Thus we may assume that A is a quasi-local integrally closed domain, π is a group of order p and pA is a radical ideal. Further $0\neq pA\neq A$, so A has characteristic zero. Since $A\pi=A[X]/(X^p-1)$ we have a natural homomorphism

$$\varphi: A\pi\to A[X]/(X-1)\times A[X]/(f)=A\times A[X]/(f)$$

where $f=X^{p-1}+X^{p-2}+\cdots+X+1$, and φ is injective since $A\pi$ is reduced. But $A\times A[X]/(f)$ is seminormal by Lemma 1.12 so it suffices by [30, Corol. 3.4] to show that $A\pi$ is seminormal in $A\times A[X]/(f)$. To this end let

$$C=(c_0, l+fA[X])\in A\times A[X]/(f)$$

be such that $C^2=\varphi(g)$ and $C^3=\varphi(h), g,h\in A\pi$. Then

$$(c_0^2, l^2+fA[X])=(g(1), g(X)+fA[X])$$

and

$$(c_0^3, l^3+fA[X])=(h(1), h(X)+fA[X]).$$

Thus $c_0^2=g(1)$, $c_0^3=h(1)$, $l^2-g\in fA[X]$, $l^3-h\in fA[X]$ and so $l^2(1)-g(1)$, $l^3(1)-h(1)\in f(1)A=pA$. But then $l(1)^2-c_0^2$ and $l(1)^3-c_0^3\in pA$ imply that $l(1)-c_0\in pA$ since A/pA is reduced [30, Lemma 3.1]. Let $l(1)=pa+c_0$, $a\in A$. Then

$$(c_0, l+fA[X])=(c_0, l-af+fA[X])=\varphi(l-af+(X^p-1)A[X]).$$

Remark 2.2. The proof of Theorem 2.1 given for Mori rings in [9] contains an error in [9, Lemma 3.2], where it is claimed that $B/q=A[X]/(f)$ is étale over A.

In [9] the conditions that $n\bar{A}$ be a radical ideal and that nA have no embedded prime divisors were not needed. This is because for Noetherian rings they both follow from the condition that nA is a radical ideal. We give a direct proof that $n\bar{A}$ is a radical ideal in nA is and A is Noetherian (the other condition being obvious). This also answers the question [9, Remark 3.4(vi)].

Theorem 2.3. *Let A be a Noetherian reduced ring, and let $p \in A$ be a regular element. If pA is a radical ideal, then $p\bar{A}$ is a radical ideal.*

Proof. Let Q be a prime ideal of $\bar{A}$ which is minimal over $p\bar{A}$, and let $P = Q \cap A$. Then P is an N-prime of pA by [23, Thm. 2.15] and hence an associated prime ideal of pA [17, 7.5]. Thus P is minimal over pA since pA is a radical ideal, and so $pA_P = PA_P$. Therefore A_P is integrally closed. Since A is reduced, we get $A_P = (\bar{A})_P$ and so Q is the only prime of $\bar{A}$ lying over P. Therefore $p\bar{A}_Q = p\bar{A}_P = pA_P = PA_P = Q\bar{A}_Q$.

We now give a partial converse to Theorem 2.1.

Proposition 2.4. *Let A be a ring and let π be a finite abelian group of order m. If $A\pi$ is seminormal, then A is seminormal, m is regular on A, and $mA, m\bar{A}$ are radical ideals.*

Proof. We can write $A\pi$ as a direct limit of seminormal subrings which are finitely generated as Z-algebras [30, Lemma 6.3], and since π is finite, these rings may be chosen of the form $A_\alpha\pi$, $A_\alpha \subsetneq A$. But then

$$\bar{A}\pi = \varinjlim \bar{A}_\alpha\pi, \qquad A/mA = \varinjlim A_\alpha/mA_\alpha, \qquad \bar{A}/m\bar{A} = \varinjlim \bar{A}_\alpha/m\bar{A}_\alpha,$$

so the conclusions follow from [9, Thm. 3.1] and Theorem 2.3.

The assumption that nA have no embedded primes in Theorem 2.1 could be removed if one knew that a B_w-prime of $pA\pi$ contracts to a B_w-prime of pA. This is not true for an arbitrary flat A algebra B [12], but it might be true for $B = A\pi$, as it is for $B = R[X]$.

Section 3

Let R be an integral domain and X an indeterminate. Following [9] we call R *quasinormal* if the canonical map $\mathrm{Pic}(R) \to \mathrm{Pic}(R[X, X^{-1}])$ is an isomorphism. (Unfortunately a different class of rings was called quasinormal in [28], [33].) In [4, Corol. 5.10] it was shown that Noetherian normal domains are quasinormal, and the Noetherian hypothesis can easily be removed via direct limits. It follows that normal $\Rightarrow$ quasinormal $\Rightarrow$ seminormal, and that neither of these implications can be reversed. Further, by [4, Thm. 6.3] the canonical maps

$$\mathrm{Pic}(R) \xrightarrow{\alpha} \mathrm{Pic}(R[X]) \xrightarrow{\beta} \mathrm{Pic}(R[X, X^{-1}])$$

are always injective and β is surjective $\Leftrightarrow$ $\beta \circ \alpha$ is surjective.

It was shown in [28, Thm. 1] that if B is an integral overring of a seminormal ring

R, then R is an intersection of rings A obtained from B by gluing over primes of R. Recall that a ring A is obtained from B by gluing over $P \in \operatorname{Spec}(R)$ if the following square is cartesian [28], [31]:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ k(P) \arrow[r] & T(B/\sqrt{PB}) \end{tikzcd} \qquad \text{where } k(P)=R_P/PR_P.$$

It is easily seen that if R is an integral domain, then $A=(R_P+\sqrt{PB_P})\cap B$.

In this section X will always be an indeterminate.

Lemma 3.1. *Let R be an integral domain and B an overring of R. If B is integral over R and R is seminormal in B then R is an intersection of rings A obtained from B by gluing over B_w-primes of principal ideals of R.*

Proof. Let $R'=\bigcap\{(R_P+\sqrt{PB_P})\cap B \mid P$ is a B_w-prime of a principal ideal of $R\}$. Then $R\subseteq R'$. If $R\neq R'$ let $b=a/c\in R'-R, a,c\in R, c\neq 0$. Then since R is seminormal in $R[b]$ the conductor $C=(R:_R R[b])$ is a radical ideal in $R[b]$ and in R. Therefore we have $C\subseteq (R:_R b)\subseteq \sqrt[R]{C}=C$. So if $P\in\operatorname{Spec}(R)$ is minimal over $(R:_R b)=(cR:_R a)$, then

$$PR_P=C_P=\sqrt{CR[b]_P}=\sqrt{PR[b]_P}.$$

Since $b\in R_P+\sqrt{PB_P}$, then

$$b=\frac{r}{s}+\frac{x}{s}, \quad r\in R,\ s\in R-P,\ x\in\sqrt{PB},$$

and thus

$$sb=r+x \ \Rightarrow\ sb-r=x\in R[b] \ \Rightarrow\ x\in R[b]\cap\sqrt{PB}=\sqrt{PR[b]}.$$

We get $b\in R_P+\sqrt{PR[b]_P}=R_P+PR_P=R_P$, contradicting $P\supseteq (R:_R b)$.

Theorem 3.2. *If R is a seminormal integral domain such that each B_w-prime of a principal ideal of R is unibranched, then R is quasinormal.*

Proof. Since $\operatorname{Pic}(K[X,X^{-1}])=0$ where K is the quotient field of R, it follows that each rank one projective module over $R[X,X^{-1}]$ is isomorphic to an ideal I of $R[X,X^{-1}]$ with $I\cap R\neq\{0\}$. Thus by Theorem 1.8 and Lemma 3.1, it suffices to show that $\operatorname{Pic}(A)\to\operatorname{Pic}(A[X,X^{-1}])$ is an isomorphism for each ring A of the form $R_P+\sqrt{P\bar{R}_P}$ where P is a minimal prime divisor of $(cR:_R a), a,c\in R$. But then A is a ring having the ideal $\sqrt{P\bar{R}_P}$ in common with the integral closure $\bar{R}_P$ of A, and since P has only one prime in $\bar{R}$ lying over it, A and $\bar{R}_P$ are quasi-local with the same maximal ideal $M=\sqrt{P\bar{R}_P}$. The result now follows from the next lemma.

Lemma 3.3. *Let R, m be a quasi-local domain and let A, m be a quasi-local overring with the same maximal ideal. If* $\mathrm{Pic}(A[X,X^{-1}])=0$, *then* $\mathrm{Pic}(R[X,X^{-1}])=0$.

Proof. Let $B=R[X,X^{-1}]$, and $S=B-mB$. We have an exact sequence [3, p. 136] $U(S^{-1}B)\to\mathrm{Pic}(B,S)\to\mathrm{Pic}(B)\to\mathrm{Pic}(S^{-1}B)$, and since $S^{-1}B=R(X)=N^{-1}R[X]$ where $N=R[X]-mR[X]$, we get from [1, Thm. 2] that $\mathrm{Pic}(S^{-1}B)=0$. Thus to show that $\mathrm{Pic}(B)=0$, it suffices to show that each invertible ideal I of B with $c(I)=R$ is principal. Let $g\in I\cap R[X]$ have minimal degree among elements of $I\cap R[X]$. To show that I is principal it suffices to show that $c(g)$ is principal. For, since B is a content R-algebra, B/I is R-flat [18, Thm. 6.5] and thus if $g=af$ for $a\in R$, $f\in B$ with $c(f)=R$, then $f\in I\cap R[X]$. Therefore if $h\in I$, then $X^nh\in I\cap R[X]$ for some n, and so $rX^nh=fq+l$ for some $r\in R, q, l\in R[X]$ and $l=0$ or degree $l<$ degree f. But since $X^nh, f\in I\cap R[X]$, we get $l=0$. Thus $rc(X^nh)=c(fq)=c(q)\Rightarrow X^nh=fq'$ some $q'\in R[X]\Rightarrow h\in fB$.

Now let $g=a_0+a_1X+\cdots+a_nX^n$ have minimal degree in $I\cap R[X]$. If $c(g)$ is not principal let $s\in\{0,1,\dots,n-1\}$ and t be such that $s+t<n$ and $c(g)$ is generated by $G=\{a_0,a_1,\dots,a_s,a_{s+t+1},a_{s+t+2},\dots,a_n\}$ and not by $G-\{a_s\}$ or $G-\{a_{s+t+1}\}$. If $t\neq 0$ write

$$a_j=\sum_{i=0}^{s} r_{ji}a_i+\sum_{i=s+t+1}^{n} r_{ji}a_i$$

for $j\in\{s+1,s+2,\dots,s+t\}$. Then

$$\begin{aligned}
g&=a_0+a_1X+\cdots+a_sX^s+\left(\sum_{i=0}^{s} r_{s+1,i}a_i+\sum_{i=s+t+1}^{n} r_{s+1,i}a_i\right)X^{s+1}\\
&\quad+\cdots+\left(\sum_{i=0}^{s} r_{s+t,i}a_i+\sum_{i=s+t+1}^{n} r_{s+t,i}a_i\right)X^{s+t}+a_{s+t+1}X^{s+t+1}+\cdots+a_nX^n\\
&=a_0\left(1+\sum_{i=s+1}^{s+t} r_{i0}X^i\right)+a_1\left(X+\sum_{i=s+1}^{s+t} r_{i1}X^i\right)\\
&\quad+\cdots+a_s\left(X^s+\sum_{i=s+1}^{s+t} r_{is}X^i\right)+a_{s+t+1}\left(X^{s+t+1}+\sum_{i=s+1}^{s+t} r_{is+t+1}X^i\right)\\
&\quad+\cdots+a_n\left(X^n+\sum_{i=s+1}^{s+t} r_{in}X^i\right).
\end{aligned}$$

Let

$$g_1=X^s+r_{s+1,s}X^{s+1}+\cdots+r_{s+t,s}X^{s+t},$$

and let x denote the image of X in B/I so that $B/I=R[x,x^{-1}]$. Since $R[x,x^{-1}]$ is R-flat and $g(x)=0$, we see that

$$\begin{aligned}
g_1(x)&\in[(a_0,a_1,\dots,a_{s-1},a_{s+t+1},a_{s+t+2},\dots,a_n)R[x,x^{-1}]:a_s]\\
&=[(a_0,a_1,\dots,a_{s-1},a_{s+t+1},\dots,a_n):a_s]R[x,x^{-1}]
\end{aligned}$$

[5, p. 47, Exer. 22]. By the choice of $t, a_s \notin (a_0, a_1, \ldots, a_{s-1}, a_{s+t+1}, \ldots, a_n)R$, so $g_1(x) \in mR[x, x^{-1}]$, and hence $g_1(X) \in I + mB$. Further, since $\mathrm{Pic}(A[X, X^{-1}]) = 0$, then $IA[X, X^{-1}]$ is principal generated by an element h of content A. It follows that h may be chosen in $IA[X, X^{-1}] \cap A[X]$ and of degree n. Then $c_A(g) = a_k A$ for some $k \in \{0, 1, \ldots, s, s+t+1, \ldots, a_n\}$, so $g = a_k \bar{g}$ where $\bar{g} = \alpha_0 + \alpha_1 X + \cdots + \alpha_n X^n$ with $\alpha_k = 1$. Further, $\alpha_s, \alpha_{s+t+1} \notin m$ for otherwise we would have

$$c_R(g) = (a_0, a_1, \ldots, a_{s-1}, a_{s+t+1}, \ldots, a_n)R$$

or

$$c_R(g) = (a_0, a_1, \ldots, a_s, a_{s+t+2}, a_{s+t+3}, \ldots, a_n)R,$$

a contradiction. Let $\varphi : A[X, X^{-1}] \to A/m[X, X^{-1}]$ be the canonical map. Then

$$g_1 \in I + mB \subseteq IA[X, X^{-1}] + mA[X, X^{-1}] \Rightarrow \varphi(g_1) \in \varphi(\bar{g})(A/m[X, X^{-1}]).$$

But this is impossible since length $\varphi(\bar{g}) \geq t+1$ and length $\varphi(g_1) \leq t$ (where length($\bar{g}$) = degree of highest degree term minus the degree of the lowest degree term of $\bar{g}$).

Corollary 3.4. *If R is a Noetherian seminormal domain such that each $P \in \mathrm{Spec}(R)$ with $\mathrm{depth}(R_P) = 1$ is unibranched, then R is quasinormal.*

The converse of the above corollary is true if R is one-dimensional [9, Thm. 4.5], but not in general [21, p. 65]. If the integral closure of R is a finite R-module, then one only needs to check that a finite set of primes is unibranched by the following:

Corollary 3.5. *If R is a Noetherian seminormal domain such that $\bar{R}$ is a finite R-module and each associated prime of $\bar{R}/R$ is unibranched, then R is quasinormal.*

Proof. This follows as in Theorem 3.3 from the fact that R is an intersection of the rings obtained from $\bar{R}$ by gluing over the associated primes of $\bar{R}/R$ [14, Thm. 1.12].

Corollary 3.6. *If R is a seminormal Hilbert domain and each maximal ideal of R is unibranched, then $\mathrm{Pic}(R) \to \mathrm{Pic}(R[X, X^{-1}])$ is an isomorphism.*

Proof. This follows since for a Hilbert ring R, if each maximal ideal is unibranched, then every prime ideal is unibranched. Indeed if say $Q_1, Q_2 \in \mathrm{Spec}(\bar{R})$ lie over $P \in \mathrm{Spec}(R)$, choose a maximal ideal M of $\bar{R}$ containing Q_1. Then $M \cap R = N$ contains P, so by the going-up theorem, there exists $M' \in \mathrm{Spec}(\bar{R})$ such that $M' \supseteq Q_2$ and $M' \cap R = N$. But since N is unibranched $M = M'$. Thus each maximal ideal which contains Q_1 contains Q_2 and so $Q_2 \subseteq Q_1$ since R is a Hilbert ring [13, Thm. 30]. Then $Q_1 \subseteq Q_2$ by symmetry.

In [11] a quasi-local domain (R, M) was defined to be a pseudo-valuation domain if for each prime ideal P of R the set $T(R) - P$ is closed under multiplication, and this was shown to be equivalent to the existence of a (unique) valuation overring V

of R with maximal ideal M. The following proposition follows from Lemma 3.3 and Theorem 1.8.

Proposition 3.7. *If an integral domain R is an intersection of pseudo-valuation overrings, then R is quasinormal.*

References

[1] D.D. Anderson, Some remarks on the ring $R(X)$, Comment. Math. Univ. St. Pauli. 26 (1977) 137–140.
[2] D.D. Anderson, J. Matijevic and W. Nichols, The Krull intersection theorem II, Pac. J. Math. 16 (1976) 15–22.
[3] H. Bass, Algebraic K-Theory (Benjamin, New York, 1968).
[4] H. Bass and M.P. Murthy, Grothendieck groups and Picard groups of abelian group rings, Ann. of Math. 86 (1967) 16–73.
[5] N. Bourbaki, Commutative Algebra (Addison-Wesley, Reading, MA, 1970).
[6] J. Brewer and W. Heinzer, Associated primes of principal ideals, Duke J. Math. 41 (1974) 1–7.
[7] S. Endo, On semi-hereditary rings, J. Math. Soc. Japan 13 (1961) 109–119.
[8] R. Gilmer and R. Heitmann, The group of units of a commutative semigroup ring, Pac. J. Math. 85 (1979) 49–64.
[9] S. Greco, Seminormality and quasinormality of group rings, J. Pure Appl. Algebra 18 (1980) 129–142.
[10] S. Greco and C. Traverso, On seminormal schemes, Compositio Mathematica 40 (1980) 325–365.
[11] J.R. Hedstrom and E.G. Houston, Pseduo-valuation domains, Pac. J. Math. 75 (1978) 137–147.
[12] W. Heinzer and J. Ohm, Locally Noetherian commutative rings, Trans. Amer. Math. Soc. 158 (1971) 273–284.
[13] I. Kaplansky, Commutative Rings (Allyn and Bacon, Boston, MA, 1970).
[14] J.V. Leahy and M.A. Vitulli, Seminormal rings and weakly normal varieties, Nagoya J. 82 (1981) 27–56.
[15] H. Matsumura, Commutative Algebra (Benjamin, New York, 1970).
[16] M.J. Marot, Une généralisation de la notion de valuation, C. R. Acad. Sci. Paris 268 (1969) 1451–1454.
[17] M. Nagata, Local Rings (Krieger, Huntington, NY, 1975).
[18] J. Ohm and D.E. Rush, Content modules and algebras, Math. Scand. 31 (1972) 49–68.
[19] T. Parker and R. Gilmer, Nilpotent elements of commutative semigroup rings, Mich. Math. J. 22 (1975) 97–108.
[20] C. Pendrini, Sul gruppo di Picard di certe estensioni di anellli di gruppo 1-dimensionali, Rend. Mat. Ser VI 4 (1971).
[21] C. Pedrini, Incollamenti e gruppi di Picard, Rend. Sem. Mat. Univ. Padova 48 (1973) 39–66.
[22] J. Querre, Idéaux divisorials d'un anneaux de polynômes, J. Algebra 64 (1980) 270–284.
[23] L.J. Ratliff, Jr., On prime divisors of the integral closure of a principal ideal, J. Reine Angew. Math. 255 (1972) 210–220.
[24] M. Raynaud, Anneaux Locaux Henséliens, Lecture Notes in Math. No. 169 (Springer, Berlin–Heidelberg–New York, 1970).
[25] P. Ribenboim, Algebraic Numbers (Wiley–Interscience, New York, 1972).
[26] D.E. Rush, The G-function of MacRae, Canad. J. Math. 26 (1974) 854–865.
[27] D.E. Rush, Content algebras, Canad. Math. Bull. 21 (1978) 329–334.
[28] D.E. Rush, Seminormality, J. Algebra 67 (1980) 377–384.

[29] R.G. Swan, Projective modules over Laurent polynomial rings, Trans. Amer. Math. Soc. 237 (1978) 111–120.

[30] R.G. Swan, On seminormality, J. Algebra 67 (1980) 210–229.

[31] C. Traverso, Seminormality and the Picard group, Ann. Scuola Norm. Sup. Pisa 24 (1970) 585–595.

[32] D. Underwood, On some uniqueness questions in primary representations of ideals, J. Math. Kyoto Univ. 9 (1969) 69–94.

[33] W.V. Vasconcelos, Quasi-normal rings, Illinois J. Math. 14 (1970) 268–273.